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INVESTIGATIONS ON CHARGED FLUID SPHERES IN GENERAL RELATIVITY

By

Dr. Jitesh Kumar, Research Scholar

University Department of Mathematics

Magadh University, Bodh Gaya

Abstract :

In this paper, we have solved Einstein-Maxwell field equations for charged perfect fluid sphere using different assumptions on metric potentials. We have also calculated and discussed physical parameters like pressure, matter density, electric field and charged density for the distribution. Further we have discussed boundary conditions using Reissner-Nordstrom metric to fix up the constants.

Key Words : Perfect fluid, charge density, matter density, electric field, pressure.

1. INTRODUCTION :

Various research workers have shown their interest in the study of charged perfect fluid distributions which has been one of the most fascinating systems in general relativity. As the Einstein-Maxwell field equations do not completely determine the system, different solutions were obtained by many authors by using different conditions to supplement the field equations. The fluid sphere of uniform density has been discussed by Kyle and Martin [14] and Mehra and Bohra [16]. Interior solutions for charged fluid sphere have also been investigated by Wilson [24], Kramer and Neugebauer [15], Krori and Barau [13], Junevicous [12] and Florides [10] by suing different conditions to supplement the field equations. The supplementary conditions were used partly to specify the physical model and partly to simplify the mathematics.

The Einstein-Maxwell field equations for spheres of charged dust have been investigated by Papapertrous [19], Bonnor and Wickramasuriya [3] and Raychaudhuri [20]. It is known that the pressure less charged distribution in equilibrium will have the absolute value of the charge to mass ratio as unity in relativistic units (De and Ray Chaudhuri [8]. A number of authors have already studied charged fluid distribution in equilibrium. Efinger [9], Bailyn and Eimerl [4] and Nduka [17, 18] have presented some solutions of charged static spherical distributions which are not free form singularity at the origin. Some exact static solutions of Einstein-Maxwell field equations representing a charged fluid sphere were obtained by Singh and Yadav [22], Shi-Chang [23] has found some conformal flat interior solutions of the Einstein Maxwell equations for a charged stable static sphere. These solutions satisfy physical conditions inside the sphere. Xingxiang [26] obtained an exact solution by specifying matter distribution and charge distribution. The metric is regular and can be matched to the Reisser-Nordstrom metric and pressure is finite. In the limit of vanishing charge, the solution can reduce to the interior solution of an uncharged sphere. Buchdahl [5] has also considered some regular general relativistic charged fluid spheres. Some other workers in this line are Glazer [11], Srivastava [23] Bonnor and Vaidya [2], Cooperstock and Cruz [7] Chakravarti and De [6], Baliyn [1], Whitman and Burch [25].

In this paper, considering spherically symmetric line element we have obtained some solutions of Einstein-Maxwell field equations using different assumptions on metric potentials. We have also discussed boundary conditions. The pressure, matter density, electric field and charge density for the distribution have been also found.

2. The Field Equations

We consider the line element in the form

(2.1) $$ds^2 = e^{\beta}dt^2 - e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where β and α are functions of r only.

The Einstein-Maxwell field equations for the charged perfect fluid distribution in general relativity are

(2.2) $$R_{\mu v} - \frac{1}{2}Rg_{\mu v} = -8\pi T_{\mu v}$$

(2.3) $$F;_{v}^{\mu v} = 4\pi J^{\mu} = 4\pi\sigma u^{\mu}$$

(2.4) $$F_{[\mu v, v]} = 0$$

where $T_{\mu v}$ is the energy momentum tensor, J^{μ} is the charged current four vector, $R_{\mu v}$ is the Ricci tensor and R the scalar of curvature tensor.

For the system under study the energy momentum tensor T_{v}^{μ} splits up into two part viz. T_{v}^{μ} and E_{v}^{μ} for matter and charges respectively.

(2.5) $$T_{v}^{\mu} = T_{v}^{\mu} + E_{v}^{\mu}$$

where

(2.6) $\bar{T}_v^{\mu} = \left[(\rho + p))u^{\mu}u_v - p\delta_{\mu}^{\mu} \right]$

with

(2.7) $u^{\mu}u_{\mu} = 1$

The non-vanishing component of T_v^{μ} are

(2.8) $\bar{T}_1^1 = T_2^2 = T_3^3 = -p$ and $T_4^4 = \rho$

Here p is internal pressure, ρ and σ are densities of matter and charges respectively, u^{μ} is the velocity vector of the matter.

The static condition is given by

(2.9) $u^1 = u^2 = u^3 = 0$ and $u^4 (g_{44})^{\frac{-1}{2}}$

i.e., $u^4 = e^{\frac{-\beta}{2}}$

The electromagnetic energy momentum tensor E_v^{μ} is given by

(2.10) $E_v^{\mu} = -F_{v\lambda}F^{\mu\lambda} + \frac{1}{4}\delta_v^{\mu}F_{lm}F^{lm}$

we assume the field to be electrostatic i.e., $F_{\mu\lambda} = 0$ and $F_{4k} = \phi_{,k} = \phi_k$, where ϕ is the electrostatic potential.

Thus the Einstein-Maxwell field equation reduces into the form

(2.11) $e^{-\alpha}\left[\frac{1}{r^2} - \frac{\alpha^1}{r}\right] - \frac{1}{r^2} = -8\pi\rho - E$

(2.12) $\frac{1}{r^2} - e^{-\alpha}\left[\frac{1}{r^2} + \frac{\beta'}{r}\right] = -8\pi p + E$

(2.13) $e^{-\alpha}\left[\frac{1}{4}\beta'\alpha' - \frac{1}{4}\beta'^2 - \frac{1}{4}\beta'' - \frac{1}{4}\left[\frac{\beta' - \alpha'}{r}\right]\right] = -8\pi p - E$

where

(2.14) $E = -F^{41}F_{41}$

and

(2.15) $4\pi\sigma = \left[\frac{\partial F^{41}}{2} + \frac{2}{r}F^{41} + \frac{\alpha' - \beta'}{2}F^{41}\right]\rho^{\beta/2}$

By the use of equations (2.11) – (2.13), we get the expressions for p, ρ and E as

(2.16) $8\pi p = \frac{e^{-\alpha}}{2}\left[\frac{\beta'}{2r} + \frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} - \frac{\alpha'}{2r} + \frac{1}{r^2}\right] - \frac{1}{2r^2}$

(2.17) $8\pi\rho = e^{-\alpha}\left[\frac{5\alpha'}{4r} - \frac{\beta''}{4} - \frac{\beta'\alpha'}{8} - \frac{\beta'^2}{8} + \frac{\beta'}{4r} - \frac{1}{2r^2}\right] + \frac{1}{2r^2}$

(2.18) $e^{-\alpha}\left[\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} - \frac{\beta'}{2r} - \frac{\alpha'}{2r} - \frac{1}{r^2}\right] + \frac{1}{r^2}$

3. Solution of The field Equations :

We have four equations (2.11) – (2.13) and (2.15) in six variables (ρ, E, p, α, β, σ). Here we take α and β two free variables. We choose

(3.1) $e^{\alpha} = \frac{k_1 r^{2n} + k}{r^{2n} + k}$

(3.2) $e^{\beta} = \frac{k_2 r^2 + k}{4k_3}$

where k_1, k_2, k_3 and k are arbitrary constants Using equation (3.1) and (3.2) in equations (2.15) – (2.18) we get

(3.5) $8\pi p = \frac{r^{2n} + k}{2(k_1 r^{2n} + k)}\left[\frac{3k_2 r^2 + 4kk_2}{(k_2 r^2 + k)}\right]$

$- \frac{(k_1 - 1)r^{2n-2}(nk)\{2k_2 r^2 + k\}}{2(k^{2n} + k)(k_1 r^{2n} + k)(k_2 r^2 + k)} + \frac{1}{2r^2}$

(3.4) $8\pi\rho = \frac{r^{2n} + k}{2(k_1 r^{2n} + k)}\left[\frac{nkr^{2n-1}(k_1 - 1)(6k_3 r^2 + 5k)}{(r^{2n} + k)(k_1 r^{2n} + k)(k_2 r^2 + k)}\right.$

$$+\frac{k_2 r^2}{2(k_1 r^{2n}+k)}-\frac{1}{r^2}\Bigg]+\frac{1}{2r^2}$$

(3.5) $$E=\frac{r^{2n}+k}{2(k_1 r^{2n}+k)}\left[\frac{-k_2^2 r^2}{(k_2 r^2+k)^2}\right.$$

$$\left.-\frac{nkr^{2n-2}(k_1-1)(2k_2 r^2+k)}{(r^{2n}+k)(k_1 r^{2n}+k)(k_2 r^2+k)}-\frac{1}{r^2}\right]+\frac{1}{2r^2}$$

(3.6) $$4\phi\sigma=\left[\frac{\partial F^{41}}{\partial r}+\left\{\frac{2}{r}+\frac{k_2 r}{k_2 r^2+k}+\frac{nkr^{2r}(k_1-1}{(r^{2n}+k)(k_1 r^{2n}+k}\right\}F^{41}\right]$$

$$\times\left[\frac{k_2 r^2+k}{4k^3}\right]^{1/2}$$

BOUNDARY CONDITIONS

We impose the following boundary conditions :

1. $e^{-\alpha}$ is is continuous across the boundary ($r = r_b$) of the fluid space.

2. The function e^{β} is continuous across the boundary ($r = r_b$) of the fluid sphere.

3. The function $\frac{de^{\beta}}{dr}$ is continuous across the boundary of the fluid sphere.

The exterior metric (i.e. for $r > r_b$) is given by Reissner-Nordstrom metric which is

(3.7) $$ds^2=\left[1-\frac{2M}{r}+\frac{Q_b^2}{r^2}\right]dt^2-\left[1-\frac{2M}{r}+\frac{Q_b^2}{r^2}\right]dr^2-r^2$$

$$(d\theta^2+\sin^2+\theta d\phi^2)$$

where $Q_b = Q_{(rb)}$ and M is the total mass of the sphere given by

(3.8) $$M=4\pi\int_0^{r_b}\rho(r),r^2dr$$

The constants appearing in the solution are fixed by the following equations :

(3.9) $$\frac{r_b^{2n}+k}{k_1 r_b^{2n}+k}=\left[1-\frac{2M}{r_b}+\frac{Q_b^2}{r_b^2}\right]$$

(3.10) $$\frac{k_2 r_b^{2n} + k}{4k_3} = \left[1 - \frac{2M}{r_b} + \frac{Q_b^2}{r_b^2}\right]$$

(3.11) $$\frac{k_2 r_b}{4k_3} = \frac{M}{r_b^2} - \frac{Q_b^2}{r_b^3}$$

Now we consider the following different cases

Case (I) : When n = 1. Then we get

(3.12) $$e^{\alpha} = \frac{k_1 r^2 + k}{r^2 + k}$$

(3.13) $$e^{\beta} = \frac{k_2 r^2 + k}{4k_3}$$

In this case, p, ρ, E and σ given by

(3.14) $$16\pi p = \frac{r^2 + k}{k_1 r^2 + k}\left[\frac{3k_2 r^2 + 4kk_2}{(k_2 r^2 + k)^2} - \frac{(k_1 - 1)(k)\{2k_2 r^2 + k\}}{(r^2 + k)(k_1 r^2 + k)(k_2 r^2 + k)} + \frac{1}{r^2}\right] - \frac{1}{r^2}$$

(3.15) $$16\pi\rho = \frac{r^2 + k}{k_1 r^2 + k}\left[\frac{k(k_1 - 1)(6k_2 r^2 + 5k)}{(r^2 + k)(k_1 r^2 + k)(k_2 r^2 + k)} + \frac{k_2^2 r^2}{(k_2 r^2 + k)^2} - \frac{1}{r^2}\right] + \frac{1}{r^2}$$

(3.16) $$E = \frac{r^2 + k}{2(k_1 r^2 + k)}\left[\frac{-k_2 r^2}{(k_2 r^2 + k)^2} - \frac{k(k_1 - 1)(2k_2 r^2 + k)}{(r^2 + k)(k_1 r^2 + k)(k_2 r^2 + k)} - \frac{1}{r^2}\right] + \frac{1}{2r^2}$$

(3.17) $$4\pi\sigma = \frac{\partial F^{41}}{\partial r} + \left\{\frac{2}{r} + \frac{k_2 r}{k_2 r^2 + k} + \frac{kr(k_1 - 1)}{(r^2 + k)(k_1 r^2 + k)}\right\}$$

$$F^{41} \times \left[\frac{k_2 r^2 + k}{4k_3} \right]^{1/2}$$

At r = 0 (i.e./ at centre) we have from above equations

(3.13) – (3.15)

(3.18) $$16\pi p_0 = \frac{4k_2 - k_1 + 1}{k}$$

(3.19) $$16\pi \rho_0 = \frac{5(k_1 - 1)}{k}$$

(3.20) $$e_0 = \frac{1 - k_1}{2k}$$

Now continuity conditions yield

(3.21) $$\frac{r_b^2 + k}{k_1 r_b^2 + k} = 1 - \frac{2M}{r_b} + \frac{Q_b^2}{r_b^2}$$

(3.22) $$\frac{k_2 r_b^2 + k}{4k_3} = 1 - \frac{2M}{r_b} + \frac{Q_b^2}{r_b^2}$$

(3.23) $$\frac{k_2 + r_b}{4k} = \frac{M}{r_b^2} - \frac{Q_b^2}{r_b^2}$$

Case II

Here we take n = 1, $k_1 = 2$. Thus we have

(3.24) $$e^{\alpha} = \frac{2r^2 + k}{r^2 + k}$$

(3.25) $$e^{\beta} = \frac{k_2 r^2 + k}{4k_3}$$

Pressure p, matter density ρ, electric field E and charge density σ in this case are given by

(3.26) $$16\pi p = \frac{r^2 + k}{2r^2 + k} \left[\frac{3k_2 r^2 + 4kk_2}{(k_2 r^2 + k)^2} \right.$$

$$-\frac{(k_1-1)k\{2k_2r^2+k\}}{(r^2+k)(2r^2+k)(k_2r^2+k}+\frac{1}{r^2}\Bigg]-\frac{1}{r^2}$$

(3.27) $$16\pi\rho=\frac{(r^2+k)}{(2r^2+k)}\Bigg[\frac{k(6k_2r^2+5k)}{(r^2+k)(2r^2+k)(k_2r^2+k)}$$

$$+\frac{k_2^2}{(2r^2+k)^2}-\frac{1}{r^2}\Bigg]+\frac{1}{r^2}$$

(2.28) $$E=\frac{r^2+k}{2(2r^2+k)}\Bigg[\frac{-k_2r^2}{(k_2r^2+k)^2}$$

$$-\frac{(k_1-1)k(2r^2+k)}{(r^2+k)(2r^2+k)(k_2r^2+k)}-\frac{1}{r^2}\Bigg]+\frac{1}{r^2}$$

(2.29) $$4\pi\sigma=\left[\frac{\partial F^{41}}{\partial r}+\left\{\frac{2}{r}+\frac{k_2r}{k_2r^2+k}+\frac{kr}{(r^2+k)(2r^2+k}\right\}F^{41}\right]$$

$$\times\left[\frac{k_2r^2+k}{4k_3}\right]^{1/2}$$

Central values of these quantities are

(3.30) $$16\pi p_0=\frac{4k_2-1}{4k_3}$$

(3.31) $$16\pi\rho_0=\frac{5}{k}$$

(3.32) $$E_0=\frac{1}{2k}$$

4. Conclusion

Clearly for P_0 to be + ve, we must have k > 0 and $K_2>\frac{1}{k}$. Again using boundary conditions, we have

(4.1) $$\frac{r_b^2+k}{2r_b^2+k}=1-\frac{2M}{r_b}+\frac{Q_b^2}{r_b^2}$$

Conditions (3.21) and (3.22) remain the same. If we fix k_2 then k, k_3 at boundary r_b can be fixed up.

Further it is believed that exact solutions of the field equations in general relativity for extended charged distribution will prove useful in the study of quantum field theory in Reimannian manifold as the question of self energy becomes answerable. As such the problem of finding exact solutions of coupled Einstein-Maxwell equations for static charged fluid sphere has attracted wide attention.

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